Q1

1

Use calculus to find

$$\int \left(2\sqrt{x} + 5x^{\frac{1}{3}}\right) dx$$

[3]

$\sqrt{x} = x^{1/2}$ $\quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$

$I = \int (2\sqrt{x} + 5x^{1/3}) \, dx$

$= \int (2x^{1/2} + 5x^{1/3}) \, dx$

$= 2 \cdot \frac{x^{3/2}}{3/2} + 5 \cdot \frac{x^{4/3}}{4/3} + c$

$\frac{1}{2} + 1 = \frac{3}{2}$

$\frac{1}{3} + 1 = \frac{4}{3}$

$\boxed{= \frac{4}{3} x^{3/2} + \frac{15}{4} x^{4/3} + c}$

Q2

2

Use calculus to find the value of

$$\int_2^4 \frac{x^3 + \sqrt[3]{x}}{2\sqrt{x}} \, dx$$

giving your answer correct to 3 significant figures.

[5]

$\sqrt{x} = x^{1/2}$ $\quad \sqrt[3]{x} = x^{1/3}$ $\quad \frac{x^a}{x^b} = x^{a-b}$

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$

$I = \int_2^4 \frac{x^3 + \sqrt[3]{x}}{2\sqrt{x}} \, dx$

$= \int_2^4 \frac{x^3 + x^{1/3}}{2x^{1/2}} \, dx$

$= \int_2^4 \left(\frac{x^3}{2x^{1/2}} + \frac{x^{1/3}}{2x^{1/2}}\right) dx$

$3 - \frac{1}{2} = \frac{5}{2}$

$\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$= \int_2^4 \left(\frac{1}{2} x^{5/2} + \frac{1}{2} x^{-1/6}\right) dx$

$= \left[\frac{1}{2} \cdot \frac{x^{7/2}}{7/2} + \frac{1}{2} \cdot \frac{x^{5/6}}{5/6}\right]_2^4$

$\frac{5}{2} + 1 = \frac{7}{2}$

$-\frac{1}{6} + 1 = \frac{5}{6}$

$= \left[\frac{1}{7} x^{7/2} + \frac{3}{5} x^{5/6}\right]_2^4$

$x^{7/2} = (\sqrt{x})^7 = x^3\sqrt{x}$

$= \left(\frac{1}{7}(4)^{7/2} + \frac{3}{5}(4)^{5/6}\right) - \left(\frac{1}{7}(2)^{7/2} + \frac{3}{5}(2)^{5/6}\right)$

$\boxed{= 17.5 \text{ (3 s.f.)}}$

Q3

3

Find the equation of the curve passing through the point (4, -8) and given by

$$y = \int \left(\frac{2}{\sqrt{x}} - x - 3\right) \mathrm{d}x$$

[4]

$\frac{1}{\sqrt{x}} = x^{-1/2}$

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$

$y = \int \left(\frac{2}{\sqrt{x}} - x - 3\right) dx$

$= \int \left(2x^{-1/2} - x - 3\right) dx$

$= 2 \cdot \frac{x^{1/2}}{1/2} - \frac{x^2}{2} - 3x + c$ $\quad -\frac{1}{2} + 1 = \frac{1}{2}$

$= 4\sqrt{x} - \frac{1}{2}x^2 - 3x + c$

When $x = 4$, $y = -8$, so

$-8 = 4\sqrt{4} - \frac{1}{2}(4)^2 - 3(4) + c$

$-8 = c - 12$

$c = 4$

$$\boxed{y = 4\sqrt{x} - \frac{1}{2}x^2 - 3x + 4}$$

Q4a

4a

(a) Show that

$$\left(3 - \frac{1}{2}x\right)^3 = 27 - \frac{27}{2}x + \frac{9}{4}x^2 - \frac{1}{8}x^3$$

[3]

(b) Hence, or otherwise, work out

$$\int \left(2\left(3 - \frac{1}{2}x\right)\right)^3 \mathrm{d}x$$

[3]

Binomial Expansion

$(a+b)^3 = {}^3C_0 a^3 + {}^3C_1 a^2 b + {}^3C_2 a b^2 + {}^3C_3 b^3$

$= a^3 + {}^3C_1 a^2 b + {}^3C_2 a b^2 + b^3$

where ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

a)

1 1

1 2 1

1 3 3 1

Pascal's Triangle

$\left(3 - \frac{1}{2}x\right)^3 = 3^3 + 3(3)^2\left(-\frac{1}{2}x\right) + 3(3)\left(-\frac{1}{2}x\right)^2 + \left(-\frac{1}{2}x\right)^3$

$= 27 + 3(9)\left(-\frac{1}{2}x\right) + 3(3)\left(\frac{1}{4}x^2\right) + \left(-\frac{1}{8}x^3\right)$

$$\boxed{= 27 - \frac{27}{2}x + \frac{9}{4}x^2 - \frac{1}{8}x^3}$$

Q4b

4b

(a) Show that

$$\left(3-\frac{1}{2}x\right)^3 = 27-\frac{27}{2}x+\frac{9}{4}x^2-\frac{1}{8}x^3$$

[3]

(b) Hence, or otherwise, work out

$$\int\left(2\left(3-\frac{1}{2}x\right)\right)^3\,dx$$

[3]

$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c$
$(n \neq -1)$

b) $I = \int\left(2\left(3-\frac{1}{2}x\right)\right)^3 dx$

$= \int\left((2)^3\left(3-\frac{1}{2}\right)^3\right) dx \qquad (ab)^n = a^n b^n$

$= \int\left(8\left(27-\frac{27}{2}x+\frac{9}{4}x^2-\frac{1}{8}x^3\right)\right) dx$

$= \int\left(216-108x+18x^2-x^3\right) dx$

$= 216x - 108\cdot\frac{x^2}{2}+18\cdot\frac{x^3}{3}-\frac{x^4}{4}+c$

$\boxed{= 216x-54x^2+6x^3-\frac{1}{4}x^4+c}$

Q5

5

Given

$$\int_q^{4q} 5x\sqrt{x}\,dx = 15\,066$$

find the value of the constant q.

[5]

$x\sqrt{x} = x\left(x^{1/2}\right) = x^{3/2}$

$x^2\sqrt{x} = x^2\left(x^{1/2}\right) = x^{5/2}$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c$
$(n \neq -1)$

$I = \int_q^{4q} 5x\sqrt{x}\,dx = \int_q^{4q} 5x^{3/2}\,dx$

$= \left[5\cdot\frac{x^{5/2}}{5/2}\right]_q^{4q}$ $\frac{3}{2}+1=\frac{5}{2}$

$= \left[2x^{5/2}\right]_q^{4q} = \left[2x^2\sqrt{x}\right]_q^{4q}$

$= \left(2(4q)^2\sqrt{4q}\right)-\left(2q^2\sqrt{q}\right)$

$= \left(64q^2\sqrt{q}\right)-\left(2q^2\sqrt{q}\right) = 62q^2\sqrt{q}$

$15066 = 62q^2\sqrt{q}$

$q^2\sqrt{q} = 243$

$q^{5/2} = 243$

$q = 243^{2/5}$ $\left(q^{5/2}\right)^{2/5} = q^{\frac{5}{2}\times\frac{2}{5}} = q$

$\boxed{q = 9}$

Q6

6

A function, $f(x)$, has second derivative given by

$$f''(x) = 2(18x - 5).$$

Given that $(2x - 1)$ and $(3x + 2)$ are factors of $f(x)$, find $f(x)$.

[6]

Factor Theorem

$f'(x) = \int f''(x)\,dx \quad (+c)$

$f(x) = \int f'(x)\,dx \quad (+d)$

From the Fundamental Theorem of Calculus

c and d are integration constants

$f'(x) = \int f''(x)\,dx = \int 2(18x-5)\,dx$

$= \int (36x - 10)\,dx$

$= 36 \cdot \frac{x^2}{2} - 10x + c = 18x^2 - 10x + c$

$f(x) = \int f'(x)\,dx = \int (18x^2 - 10x + c)\,dx$

$= 18 \cdot \frac{x^3}{3} - 10 \cdot \frac{x^2}{2} + cx + d$

$= 6x^3 - 5x^2 + cx + d$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{2}c + d = 0$

$c + 2d = 1$ ①

$f\left(-\frac{2}{3}\right) = -4 - \frac{2}{3}c + d = 0$

$-2c + 3d = 12$ ②

simultaneous equations

$2 \times ① + ②$: $7d = 14 \Rightarrow d = 2 \Rightarrow c = -3$

$$\boxed{f(x) = 6x^3 - 5x^2 - 3x + 2}$$

Q7a

7a

(a) Given that

$$\int_p^{\infty} \frac{3}{x\sqrt{x}}\,dx = \sqrt{3}$$

Improper integral!

where p is a real constant, find the value of p.

[5]

(b) Given that

$$\int_0^{50} \frac{q + 3x}{\sqrt{x}}\,dx = 750\sqrt{2}$$

where q is a real constant, find the value of q.

[5]

Replace ∞ with a and integrate

a) $\int_p^a \frac{3}{x\sqrt{x}}\,dx = \int_p^a 3x^{-3/2}\,dx$

$= \left[3\left(\frac{1}{-1/2} x^{-1/2}\right)\right]_p^a$

$= \left[-6x^{-1/2}\right]_p^a$

$= \left[-\frac{6}{\sqrt{x}}\right]_p^a$

$= \frac{6}{\sqrt{p}} - \frac{6}{\sqrt{a}}$

Then take limit as $a \to \infty$

$\lim_{a\to\infty} \frac{6}{\sqrt{a}} = 0$

So $\int_p^{\infty} \frac{3}{x\sqrt{x}}\,dx = \lim_{a\to\infty}\left(\frac{6}{\sqrt{p}} - \frac{6}{\sqrt{a}}\right) = \frac{6}{\sqrt{p}} - 0 = \frac{6}{\sqrt{p}}$

Therefore

$\frac{6}{\sqrt{p}} = \sqrt{3} \Rightarrow \sqrt{p} = \frac{6}{\sqrt{3}} \Rightarrow p = \frac{36}{3}$

$$\boxed{p = 12}$$

Q7b

7b

(a) Given that

$$\int_p^{\infty} \frac{3}{x\sqrt{x}}\,dx = \sqrt{3}$$

where p is a real constant, find the value of p.

[5]

(b) Given that

$$\int_0^{50} \frac{q+3x}{\sqrt{x}}\,dx = 750\sqrt{2}$$

where q is a real constant, find the value of q.

[5]

This is an improper integral, because $\frac{q+3x}{\sqrt{x}}$ isn't defined when $x=0$.

Replace 0 with a and integrate

b) $\int_a^{50} \frac{q+3x}{\sqrt{x}}\,dx = \int_a^{50}\left(\frac{q}{\sqrt{x}} + \frac{3x}{\sqrt{x}}\right)dx = \int_a^{50}\left(qx^{-1/2} + 3x^{1/2}\right)dx$

$= \left[2qx^{1/2} + 2x^{3/2}\right]_a^{50} = \left[2q\sqrt{x} + 2x\sqrt{x}\right]_a^{50}$

$= \left(2q\sqrt{50} + 2(50)\sqrt{50}\right) - \left(2q\sqrt{a} + 2a\sqrt{a}\right)$

$= \left(10q\sqrt{2} + 500\sqrt{2}\right) - \left(2q\sqrt{a} + 2a\sqrt{a}\right)$

Then take limit as $a \to 0$

So $\int_0^{50} \frac{q+3x}{\sqrt{x}}\,dx$

$= \lim_{a\to 0}\left(\left(10q\sqrt{2} + 500\sqrt{2}\right) - \left(2q\sqrt{a} + 2a\sqrt{a}\right)\right)$

$= \left(10q\sqrt{2} + 500\sqrt{2}\right) - (0+0)$

$= 10q\sqrt{2} + 500\sqrt{2} = (10q+500)\sqrt{2}$

Therefore

$(10q+500)\cancel{\sqrt{2}} = 750\cancel{\sqrt{2}}$

$10q + 500 = 750 \Rightarrow 10q = 250$

$\boxed{q = 25}$